Well-posedness for the heat flow of biharmonic maps with rough initial data

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Abstract

This paper establishes the local (or global, resp.) well-posedness of the heat flow of bihharmonic maps from $\mathbb{R}^n$ to a compact Riemannian manifold without boundary for initial data with small local BMO (or BMO, resp.) norms.

1 Introduction

For $k \geq 1$, let N be a k-dimensional compact Riemannian manifold without boundary, isometrically embedded in some Euclidean space $\mathbb{R}^l$. Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a smooth domain. There are two second order energy functional for mappings from Ω to N, namely, the Hessian energy functional and tension field energy functional given by

$$F(u) = \int_\Omega |\Delta u|^2,\ E(u) = \int_\Omega |D\Pi(u)(\Delta u)|^2,\ u \in W^{2,2}(\Omega, N),$$

where $\Pi : N_{\delta_N} \to N$ is the smooth nearest point projection from $N_{\delta_N} = \{y \in \mathbb{R}^l : \mathrm{dist}(y, N) \leq \delta_N\}$ to N for some small $\delta_N > 0$, and

$$W^{2,2}(\Omega, N) = \{v \in W^{2,2}(\Omega, \mathbb{R}^l) :\ v(x) \in N \text{ for a.e. } x \in \Omega\}.$$

Recall that a map $u \in W^{2,2}(\Omega, N)$ is called an (extrinsic) biharmonic map (or intrinsic biharmonic map, resp.) if u is a critical point of $F(\cdot)$ (or $E(\cdot)$, resp.). Geometrically, a biharmonic map u to N enjoys the property that $\Delta^2 u$ is perpendicular

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to T_uN. The Euler-Lagrange equation for biharmonic maps (see [17]) is:

$$\Delta^2 u = \Delta(D^2\Pi(u)(\nabla u, \nabla u)) + 2\nabla\cdot\langle\Delta u, \nabla(D\Pi(u))\rangle - \langle\Delta u, \Delta(D\Pi(u))\rangle. \tag{1.1}$$

The Euler-Lagrange equation for intrinsic biharmonic maps (see [17]) is:

$$\begin{aligned}\Delta^2 u &= \Delta(D^2\Pi(u)(\nabla u, \nabla u)) + 2\nabla\cdot\langle\Delta u, \nabla(D\Pi(u))\rangle - \langle\Delta u, \Delta(D\Pi(u))\rangle\\ &+ D\Pi(u)[D^2\Pi(u)(\nabla u, \nabla u)\cdot D^3\Pi(u)(\nabla u, \nabla u)]\\ &+ 2D^2\Pi(u)(\nabla u, \nabla u)\cdot D^2\Pi(u)(\nabla u, \nabla(D\Pi(u))).\end{aligned} \tag{1.2}$$

The study of biharmonic maps was initiated by Chang-Wang-Yang [2] in late 90's. It has since drawn considerable research interests. In particular, the smoothness of biharmonic maps (and intrinsic biharmonic maps) in $W^{2,2}$ has been established in dimension 4 by [2] for $N = S^{l-1}$ and by [16] for general manifold N. For $n \geq 5$, the partial regularity of the class of stationary biharmonic maps in $W^{2,2}$ has been shown by by [2] for $N = S^{l-1}$ and by [16] for general manifold N. The readers can refer to Strzelecki [15], Angelesberg [1], Lamm-Riviere [11], Struwe [14], Scheven [12], Hong-Wang [4], and Wang [18] for further interesting results.

Motivated by the study of heat flow of harmonic maps, which has played a very important role in the existence of harmonic maps in various topological classes, it is very natural and interesting to study the corresponding heat flow of biharmonic maps. For $\Omega = \mathbb{R}^n$, the heat flow of harmonic maps for $u : \mathbb{R}^n \times \mathbb{R}_+ \to N$ is given by

$$\begin{aligned}\partial_t u + \Delta^2 u &= \Delta(D^2\Pi(u)(\nabla u, \nabla u)) + 2\nabla\cdot\langle\Delta u, \nabla(D\Pi(u))\rangle\\ &\quad -\langle\nabla\Delta u, \Delta(D\Pi(u))\rangle \qquad \text{in } \mathbb{R}^n\times(0,+\infty) \qquad (1.3)\\ u|_{t=0} &= u_0 \qquad \text{on } \mathbb{R}^n, \qquad (1.4)\end{aligned}$$

where $u_0 : \mathbb{R}^n \to N$ is a given map.

(1.3)-(1.4) was first investigated by Lamm in [8, 9], where for smooth initial data $u_0 \in C^\infty(\mathbb{R}^n, N)$ the short time smooth solution was established. Moreover, such a short time smooth solution is proven to be globally smooth provided that $n = 4$ and $\|u_0\|_{W^{2,2}(\mathbb{R}^4)}$ is sufficiently small. For large initial data $u_0 \in W^{2,2}(\mathbb{R}^4)$, it was

independently proved by Gastel [3] and Wang [19] that there exists a global weak solution to (1.3)-(1.4) that is smooth away from finitely many singular times.

It is a very interesting question to seek the largest class of rough initial data such that (1.3)-(1.4) is well-posed (either local or global) in suitable spaces. There have been interesting works on this type of question for the Navier-Stokes equation (see Koch-Tataru [7]), the heat flow of harmonic maps (see Koch-Lamm [6] and Wang [20]), and the Willmore flow, the Ricci flow, and the Mean curvature flow by Koch-Lamm [6].

The main goal of this paper is to investigate the well-posedness issue of (1.3) and (1.4) for initial data u_0 with small BMO norm.

To state our main result, we first introduce the BMO spaces.

Definition 1.1 *For $0 < R \leq +\infty$, the local BMO space, $\mathrm{BMO}_R(\mathbb{R}^n)$, is the space consisting of locally integrable functions f such that*

$$[f]_{\mathrm{BMO_R}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n, 0<r\leq R} \{r^{-n} \int_{B_r(x)} |f - f_{x,r}|\} < +\infty,$$

where $B_r(x) \subset \mathbb{R}^n$ is the ball with center x and radius r, and

$$f_{x,r} = \frac{1}{|B_r(x)|} \int_{B_r(x)} f$$

is the average of f over $B_r(x)$. We say $f \in \overline{\mathrm{VMO}}(\mathbb{R}^n)$ if

$$\lim_{r\downarrow 0} [f]_{\mathrm{BMO}_r(\mathbb{R}^n)} = 0.$$

For $R = +\infty$, we simply write $(\mathrm{BMO}(\mathbb{R}^n), [\cdot]_{\mathrm{BMO}(\mathbb{R}^n)})$ for $(\mathrm{BMO}_\infty(\mathbb{R}^n), [\cdot]_{\mathrm{BMO}_\infty(\mathbb{R}^n)})$.

For $0 < T \leq +\infty$, we also introduce the functional space X_T as follows.

$$X_T = \left\{ f : \mathbb{R}^n \times [0, T] \to \mathbb{R} \mid \|f\|_{X_T} \equiv \sup_{0<t\leq T} \|f(t)\|_{L^\infty(\mathbb{R}^n)} + [f]_{X_T} < +\infty \right\} \quad (1.5)$$

where

$$\begin{aligned} [f]_{X_T} \quad &= \quad \sup_{0<t\leq T} (\sum_{i=1}^{2} t^{\frac{i}{4}} \|\nabla^i f(t)\|_{L^\infty(\mathbb{R}^n)}) + \sup_{x\in\mathbb{R}^n, 0<R\leq T^{\frac{1}{4}}} (R^{-n} \int_{P_R(x,R^4)} |\nabla f|^4)^{\frac{1}{4}} \\ &+ \sup_{x\in\mathbb{R}^n, 0<R\leq T^{\frac{1}{4}}} (R^{-n} \int_{P_R(x,R^4)} |\nabla^2 f|^2)^{\frac{1}{2}}, \end{aligned} \quad (1.6)$$

where $P_R(x, R^4) = B_R(x) \times [0, R^4]$ is the parabolic cylinder with center (x, R^4) and radius R. It is clear that $(X_T, \|\cdot\|_{X_T})$ is a Banach space. When $T = +\infty$, we simply write X for X_∞, $\|\cdot\|_X$ for $\|\cdot\|_{X_\infty}$, and $[\cdot]_X$ for $[\cdot]_{X_\infty}$.

The first theorem states

Theorem 1.2 *There exists $\epsilon_0 > 0$ such that for any $R > 0$ if $[u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)} \leq \epsilon_0$, then there exists a unique solution $u \in X_{R^4}$ to (1.3)-(1.4) with small $[u]_{X_T}$. In particular, if $u_0 \in \overline{\mathrm{VMO}}(\mathbb{R}^n)$ then there exists $T_0 > 0$ such that (1.3)-(1.4) admits a unique solution $u \in X_{T_0}$ with small $[u]_{X_{T_0}}$.*

As a direct corollary, we have the following global well-posedness result.

Theorem 1.3 *There exists $\epsilon_0 > 0$ such that if $[u_0]_{\mathrm{BMO}(\mathbb{R}^n)} \leq \epsilon_0$, then there exists a unique solution $u \in X$ to (1.3)-(1.4) with small $[u]_X$.*

Now we turn to the discussion of the heat flow of intrinsic biharmonic maps. The equation of the heat flow of intrinsic biharmonic maps on $\mathbb{R}^n$ is given by

$$\begin{aligned}
\partial_t u + \Delta^2 u &= \Delta(D^2\Pi(u)(\nabla u, \nabla u)) + 2\nabla \cdot \langle \Delta u, \nabla(D\Pi(u))\rangle - \langle \Delta u, \Delta(D\Pi(u))\rangle \\
&+ D\Pi(u)[D^2\Pi(u)(\nabla u, \nabla u) \cdot D^3\Pi(u)(\nabla u, \nabla u)] \\
&+ 2D^2\Pi(u)(\nabla u, \nabla u) \cdot D^2\Pi(u)(\nabla u, \nabla(D\Pi(u)) \text{ in } \mathbb{R}^n \times (0, +\infty) \quad (1.7)\\
u\big|_{t=0} &= u_0 : \mathbb{R}^n \to N. \quad (1.8)
\end{aligned}$$

In [10], Lamm studied (1.7)-(1.8). Under the assumption that $n \leq 4$ and the section curvature of N is nonpositive, the global smooth solution to (1.7)-(1.8) was established in [10].

Analogous to Theorem 1.2 and 1.3, we obtain the following results on (1.7)-(1.8).

Theorem 1.4 *There exists $\epsilon_0 > 0$ such that for any $R > 0$ if $[u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)} \leq \epsilon_0$, then there exists a unique solution $u \in X_{R^4}$ to (1.7)-(1.8) with small $[u]_{X_T}$. In particular, if $u_0 \in \overline{\mathrm{VMO}}(\mathbb{R}^n)$ then there exists $T_0 > 0$ such that (1.7)-(1.8) admits a unique solution $u \in X_{T_0}$ with small $[u]_{X_{T_0}}$.*

Theorem 1.5 *There exists $\epsilon_0 > 0$ such that if $[u_0]_{\mathrm{BMO}(\mathbb{R}^n)} \leq \epsilon_0$, then there exists a unique solution $u \in X$ to (1.7)-(1.8) with small $[u]_X$.*

We remark that since $W^{1,n}(\mathbb{R}^n) \subset \overline{\mathrm{VMO}}(\mathbb{R}^n)$, it follows from Theorem 1.2 (or Theorem 1.4, resp.) that (1.3)-(1.4) (or 1.7)-(1.8), resp.) is uniquely solvable in X_{T_0} for some $T_0 > 0$ provided $u_0 \in W^{1,n}(\mathbb{R}^n, N)$; and is uniquely solvable in X provided $\|\nabla u_0\|_{L^n(\mathbb{R}^n)}$ is sufficiently small, via Theorem 1.3 (or Theorem 1.5, resp.).

We also remark that the techniques to handle the heat flow of biharmonic maps illustrated in this paper can be extended to investigate the well-posedness of the heat flow of polyharmonic maps for BMO initial data in any dimensions. This will be discussed in a forthcoming paper [5].

The remaining of the paper is written as follows. In section 2, we review some basic estimates on the biharmonic heat kernel, due to Koch-Lamm [6]. In section 3, we outline some crucial estimates on the biharmonic heat equation. In section 4, we prove the boundedness of the mapping operator $\mathbb{S}$ determined by the Duhamel formula. In section 5, we prove Theorem 1.2 and 1.3. In section 6, we prove Theorem 1.4 and 1.5.

2 Review of the biharmonic heat kernel

In this section, we review some fundamental properties from Koch and Lamm [6] on the biharmonic heat kernel.

Consider the fundamental solution of the biharmonic heat equation:

$$(\partial_t + \Delta^2)b(x,t) = 0 \text{ in } \mathbb{R}^n \times \mathbb{R}_+$$

and it is given by

$$b(x,t) = t^{-\frac{n}{4}} g(\frac{x}{t^{\frac{1}{4}}}),$$

where

$$g(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\xi k - |k|^4}\, dk,\ \xi \in \mathbb{R}^n. \tag{2.1}$$

The following Lemma, due to Koch and Lamm [6] (Lemma 2.4), play a very important role in this paper.

Lemma 2.1 *For $x \in \mathbb{R}^n$ and $t > 0$, the following estimates hold:*

$$|b(x,t)| \le ct^{-\frac{n}{4}} \exp(-\alpha \frac{|x|^{\frac{4}{3}}}{t^{\frac{1}{3}}}),\ \alpha = \frac{32^{\frac{1}{3}}}{16}, \tag{2.2}$$

$$|\nabla^k b(x,t)| \le c(t^{\frac{1}{4}} + |x|)^{-n-k},\ \forall k \ge 1 \tag{2.3}$$

$$\|\nabla^k b(\cdot,t)\|_{L^1(\mathbb{R}^n)} \le ct^{-\frac{k}{4}},\ \forall k \ge 1. \tag{2.4}$$

Moreover, there exist $c, c_1 > 0$ such that for $0 \le j \le 4$,

$$|\nabla^j b(x,t)| \le ce^{-c_1|x|},\ \forall (x,t) \in \mathbb{R}^n \times (0,1) \setminus (B_2 \times (0, \frac{1}{2})). \tag{2.5}$$

For the purpose of this paper, we also recall the Carleson's characterization of BMO spaces. Let $\mathcal{S}$ denote the class of Schwartz functions. Then the following property is well-known (see, Stein [13]).

Lemma 2.2 *Let $\Phi \in \mathcal{S}$ be such that $\int_{\mathbb{R}^n} \Phi = 0$. For $t > 0$, let $\Phi_t(x) = t^{-n}\Phi(\frac{x}{t})$, $x \in \mathbb{R}^n$. If $f \in \mathrm{BMO}(\mathbb{R}^n)$, then $|\Phi_t * f|^2(x,t)\frac{dxdt}{t}$ is a Carleson measure on $\mathbb{R}^{n+1}_+$, i.e.,*

$$\sup_{x\in\mathbb{R}^n, r>0} r^{-n} \int_0^r \int_{B_r(x)} |\Phi_t * f|^2 \frac{dxdt}{t} \le C[u_0]^2_{\mathrm{BMO}(\mathbb{R}^n)} \tag{2.6}$$

for some $C = C(n) > 0$. If $f \in \mathrm{BMO}_R(\mathbb{R}^n)$ for some $R > 0$, then

$$\sup_{x\in\mathbb{R}^n, 0<r\le R} r^{-n} \int_0^r \int_{B_r(x)} |\Phi_t * f|^2 \frac{dxdt}{t} \le C[u_0]^2_{\mathrm{BMO}_R(\mathbb{R}^n)} \tag{2.7}$$

for some $C = C(n) > 0$.

Recall that the solution to the Dirichlet problem of the inhomogeneous biharmonic heat equation

$$(\partial_t + \Delta^2)u = f \text{ on } \mathbb{R}^n \times (0, +\infty) \tag{2.8}$$

$$u = u_0 \text{ on } \mathbb{R}^n \times \{0\} \tag{2.9}$$

is given by the Duhamel formula:

$$u = \mathbb{G}u_0 + \mathbb{S}f \tag{2.10}$$

where

$$\mathbb{G}u_0(x,t) := (b(\cdot,t) * u_0)(x) = \int_{\mathbb{R}^n} b(x-y,t)u_0(y)\,dy,\ (x,t) \in \mathbb{R}^n \times (0,+\infty), \tag{2.11}$$

and

$$\mathbb{S}f(x,t) = \int_0^t \int_{\mathbb{R}^n} b(x-y,t-s)f(y,s)\,dyds,\ (x,t) \in \mathbb{R}^n \times (0,+\infty). \tag{2.12}$$

3 Basic estimates for the biharmonic heat equation

In this section, we provide some crucial estimates for the solution of the biharmonic heat equation with initial data in BMO spaces, including the estimate of the distance to the manifold N.

Lemma 3.1 *For $0 < R \le +\infty$, if $u_0 \in \mathrm{BMO}_R(\mathbb{R}^n)$, then $\hat{u}_0 \equiv \mathbb{G}u_0$ satisfies the following estimates:*

$$\sup_{x\in\mathbb{R}^n, 0<r\le R} r^{-n}\int_{P_r(x,r^4)} (|\nabla^2\hat{u}_0|^2 + r^{-2}|\nabla\hat{u}_0|^2) \le C\,[u_0]^2_{\mathrm{BMO}_R(\mathbb{R}^n)}\,, \tag{3.1}$$

and

$$\sup_{0<t\le R^4}\left(\sum_{i=1}^{2} t^{\frac{i}{4}}\|\nabla\hat{u}_0(t)\|_{L^\infty(\mathbb{R}^n)}\right) \le C\,[u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)}\,. \tag{3.2}$$

If, in addition, $u_0 \in L^\infty(\mathbb{R}^n)$, then

$$\sup_{x\in\mathbb{R}^n, 0<r\le R} r^{-n}\int_{P_r(x,r^4)} |\nabla\hat{u}_0|^4 \le C\|u_0\|^2_{L^\infty(\mathbb{R}^n)}\cdot[u_0]^2_{\mathrm{BMO}_R(\mathbb{R}^n)}\,. \tag{3.3}$$

Proof. For simplicity, we present the argument for $R = +\infty$. Let g be given by (2.1). Let $\Phi^i = \nabla^i g$ for $i = 1, 2$. Then it is clear that $\Phi^i \in \mathcal{S}$ and $\int_{\mathbb{R}^n}\Phi^i = 0$ for $i = 1, 2$. Hence by Lemma 2.2, $|\Phi^i_t * u_0|^2\frac{dxdt}{t}$ is a Carleson measure on $\mathbb{R}^{n+1}_+$ for $i = 1, 2$. Direct calculations show, for $i = 1, 2$,

$$\Phi^i_t(x) = t^{-n}(\nabla^i g)(\frac{x}{t}) = t^i\nabla^i\left(t^{-n}g(\frac{x}{t})\right) = t^i\nabla^i(g_t(x)),$$

where

$$g_t(x) = t^{-n}g(\frac{x}{t}).$$

Hence we have

$$(\Phi^i_t * u_0)(x) = t^i\nabla^i(g_t * u_0)(x).$$

Since the biharmonic heat kernel $b(x,t) = g_{t^{\frac{1}{4}}}(x)$, we have

$$(\Phi^i_t * u_0)(x) = t^i\nabla^i\left((b(\cdot,t^4) * u_0)(x)\right) = t^i\nabla^i(\mathbb{G}u_0)(x,t^4).$$

Thus we have, for $i = 1, 2$,

$$\begin{aligned}
C[u_0]^2_{\mathrm{BMO}(\mathbb{R}^n)} &\ge \sup_{x\in\mathbb{R}^n, r>0} r^{-n}\int_0^r\int_{B_r(x)} |\Phi^i_t * u_0|^2\frac{dxdt}{t}\\
&= \sup_{x\in\mathbb{R}^n, r>0} r^{-n}\int_0^r\int_{B_r(x)} t^{2i-1}|\nabla^i\mathbb{G}u_0|^2(x,t^4)\,dxdt\\
&= \frac{1}{4}\sup_{x\in\mathbb{R}^n, r>0} r^{-n}\int_{P_r(x,r^4)} t^{\frac{2i-4}{4}}|\nabla^i\mathbb{G}u_0|^2(x,t)\,dxdt
\end{aligned}$$

This clearly implies (3.1), since for $i = 1, 2$, $t^{\frac{2i-4}{4}} \geq r^{2i-4}$ when $0 \leq t \leq r^4$.

Since $\hat{u}_0$ solves the biharmonic heat equation $(\partial_t + \Delta^2)\hat{u}_0 = 0$ on $\mathbb{R}^n \times (0, +\infty)$, the standard gradient estimate implies that for any $x \in \mathbb{R}^n$ and $r > 0$,

$$r^2|\nabla \hat{u}_0|^2(x, r^4) + r^4|\nabla^2 \hat{u}_0|^2(x, r^4) \leq Cr^{-n} \int_{P_r(x,r^4)} (r^{-2}|\nabla \hat{u}_0|^2 + |\nabla^2 \hat{u}_0|^2).$$

Taking supremum over $x \in \mathbb{R}^n$ and setting $t = r^4 > 0$ yields (3.2).

For (3.3), observe that $u_0 \in L^\infty(\mathbb{R}^n)$ implies $\Phi^1_t * u_0 \in L^\infty(\mathbb{R}^n)$ and

$$\|\Phi^1_t * u_0\|_{L^\infty(\mathbb{R}^n)} \leq \|\Phi^1\|_{L^1(\mathbb{R}^n)}\|u_0\|_{L^\infty(\mathbb{R}^n)} \leq \|\nabla g\|_{L^1(\mathbb{R}^n)}\|u_0\|_{L^\infty(\mathbb{R}^n)} \leq C\|u_0\|_{L^\infty(\mathbb{R}^n)}.$$

Hence

$$\begin{aligned}
& \sup_{x\in\mathbb{R}^n, r>0} \int_{P_r(x,r^4)} |\nabla \mathbb{G}u_0|^4 \, dxdt \\
= \; & \sup_{x\in\mathbb{R}^n, r>0} \int_0^r \int_{B_r(x)} |\Phi^1_t * u_0|^4 \frac{dxdt}{t} \\
\leq \; & \left(\sup_{t>0} \|\Phi^1_t * u_0\|_{L^\infty(\mathbb{R}^n)} \right) \cdot \sup_{x\in\mathbb{R}^n, r>0} \int_0^r \int_{B_r(x)} |\Phi^1_t * u_0|^2 \frac{dxdt}{t} \\
\leq \; & C\|u_0\|^2_{L^\infty(\mathbb{R}^n)} \cdot [u_0]^2_{\mathrm{BMO}(\mathbb{R}^n)} \,.
\end{aligned}$$

This implies (3.3). □

Now we prove an important estimate on the distance of $\hat{u}_0$ to the manifold N in terms of the BMO norm of u_0. More precisely,

Lemma 3.2 *For any $\delta > 0$, there exists $K = K(\delta, N) > 0$ such that for $R > 0$ if $u_0 \in \mathrm{BMO}_R(\mathbb{R}^n)$ then*

$$\mathrm{dist}(\hat{u}_0(x,t), N) \leq K\,[u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)} + \delta, \quad \forall x \in \mathbb{R}^n, \; 0 \leq t \leq \frac{R^4}{K^4}. \tag{3.4}$$

In particular, if $u_0 \in \mathrm{BMO}(\mathbb{R}^n)$ then

$$\mathrm{dist}(\hat{u}_0(x,t), N) \leq K\,[u_0]_{\mathrm{BMO}(\mathbb{R}^n)} + \delta, \quad \forall x \in \mathbb{R}^n, \; t \in \mathbb{R}_+. \tag{3.5}$$

Proof. Since (3.5) follows directly from (3.4), it suffices to prove (3.4). For any $x \in \mathbb{R}^n$, $t > 0$, and $K > 0$, denote

$$c^K_{x,t} = \frac{1}{|B_K(0)|} \int_{B_K(0)} u_0(x - t^{\frac{1}{4}}z)\, dz.$$

Let g be given by (2.1). Then, by a change of variables, we have

$$\hat{u}_0(x,t) = \int_{\mathbb{R}^n} g(y) u_0(x - t^{\frac{1}{4}} y)\, dy.$$

Applying Lemma 2.1, we have

$$\begin{aligned}
\left|\hat{u}_0(x,t) - c^K_{x,t}\right| &\le \int_{\mathbb{R}^n} g(y) |u_0(x - t^{\frac{1}{4}} y) - c^K_{x,t}|\, dy \\
&\le \left\{ \int_{B_K(0)} + \int_{\mathbb{R}^n \setminus B_K(0)} \right\} g(y) |u_0(x - t^{\frac{1}{4}} y) - c^K_{x,t}|\, dy \\
&\le \int_{B_K(0)} c e^{-\alpha |y|^{\frac{4}{3}}} |u_0(x - t^{\frac{1}{4}} y) - c^K_{x,t}|\, dy \\
&\quad + 2\|u_0\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n \setminus B_K(0)} c e^{-\alpha |y|^{\frac{4}{3}}}\, dy \\
&\le K^n\, [u_0]_{\mathrm{BMO}_{Kt^{\frac{1}{4}}}(\mathbb{R}^n)} + C_N \int_K^\infty e^{-\alpha r^{\frac{4}{3}}} r^{n-1}\, dr \\
&\le \delta + K^n\, [u_0]_{\mathrm{BMO}_{Kt^{\frac{1}{4}}}(\mathbb{R}^n)} \qquad (3.6)
\end{aligned}$$

provide we choose a sufficiently large $K = K(\delta, N) > 0$ so that

$$C_N \int_K^\infty e^{-\alpha r^{\frac{4}{3}}} r^{n-1}\, dr \le \delta.$$

On the other hand, since $u_0(\mathbb{R}^n) \subset N$, we have

$$\mathrm{dist}(c^K_{x,t}, N) \le \left| c^K_{x,t} - u_0(x - t^{\frac{1}{4}} y) \right|, \ \forall y \in B_K(0)$$

and hence

$$\mathrm{dist}(c^K_{x,t}, N) \le \frac{1}{|B_K(0)|} \int_{B_K(0)} |c^K_{x,t} - u_0(x - t^{\frac{1}{4}} y)|\, dy \le [u_0]_{\mathrm{BMO}_{Kt^{\frac{1}{4}}}(\mathbb{R}^n)}. \qquad (3.7)$$

Putting (3.6) and (3.7) together yields (3.4) holds for $t \le \frac{R^4}{K^4}$. This completes the proof. □

4 Boundedness of the operator $\mathbb{S}$

In this section, we introduce two more functional spaces and establish the boundedness of the operator $\mathbb{S}$ between these spaces.

For $0 < T \le +\infty$, besides the space X_T introduced in the section 1, we need to introduce the spaces Y^1_T, Y^2_T.

The space Y_T^1 is the space consisting of functions $f:\mathbb{R}^n\times[0,T]\to\mathbb{R}$ such that

$$\|f\|_{Y_T^1}\equiv\sup_{0<t\le T} t\|f(t)\|_{L^\infty(\mathbb{R}^n)}+\sup_{x\in\mathbb{R}^n,0<r\le T^{\frac14}} r^{-n}\int_{P_r(x,r^4)}|f|<+\infty, \tag{4.1}$$

and the space Y_T^2 is the space consisting of functions $f:\mathbb{R}^n\times[0,T]\to\mathbb{R}$ such that

$$\|f\|_{Y_T^2}\equiv\sup_{0<t\le T} t^{\frac34}\|f(t)\|_{L^\infty(\mathbb{R}^n)}+\sup_{x\in\mathbb{R}^n,0<r\le T^{\frac14}} (r^{-n}\int_{P_r(x,r^4)}|f|^{\frac43})^{\frac34}<+\infty. \tag{4.2}$$

It is easy to see $(Y_T^i,\|\cdot\|_{Y_T^i})$ is a Banach space for $i=1,2$. When $T=+\infty$, we simply denote $(Y^i,\|\cdot\|_{Y^i})$ for $(Y_\infty^i,\|\cdot\|_{Y_\infty^i})$ for $i=1,2$.

Let the operator $\mathbb{S}$ be defined by (2.12). Then we have

Lemma 4.1 *For any $0<T\le+\infty$, if $f\in Y_T^1$, then $\mathbb{S}f\in X_T$ and*

$$\|\mathbb{S}f\|_{X_T}\le C\|f\|_{Y_T^1} \tag{4.3}$$

for some $C=C(n)>0$.

Proof. We need to show the pointwise estimate

$$\sum_{i=0}^{2} R^i|\nabla^i(\mathbb{S}f)|(x,R^4)\le C\|f\|_{Y_T^1},\ \forall x\in\mathbb{R}^n, 0<R\le T^{\frac14}, \tag{4.4}$$

and the integral estimate for $0<R\le T^{\frac14}$:

$$R^{-\frac n4}\|\nabla(\mathbb{S}f)\|_{L^4(P_R(x,R^4))}+R^{-\frac n2}\|\nabla^2(\mathbb{S}f)\|_{L^2(P_R(x,R^4))}\le C\|f\|_{Y_T^1}. \tag{4.5}$$

By suitable scalings, we may assume $T\ge1$. Since both estimates are translation and scale invariant, it suffices to show that both (4.4) and (4.5) hold for $x=0$ and $R=1$.

For $i=0,1,2$, we have

$$\begin{aligned}\left|\nabla^i\mathbb{S}f(0,1)\right| &= \left|\int_0^1\int_{\mathbb{R}^n}\nabla^i b(y,1-s)f(y,s)\,dyds\right|\\ &\le \left\{\int_{\frac12}^1\int_{\mathbb{R}^n}+\int_0^{\frac12}\int_{B_2}+\int_0^{\frac12}\int_{\mathbb{R}^n\setminus B_2}\right\}|\nabla^i b(y,1-s)||f(y,s)|\,dyds\\ &= I_1+I_2+I_3.\end{aligned}$$

Applying Lemma 2.1, we can estimate I_1, I_2, I_3 as follows.

$$
\begin{aligned}
|I_1| &\leq \left(\sup_{\frac{1}{2}\leq s\leq 1}\|f(s)\|_{L^\infty(\mathbb{R}^n)}\right)\left(\int_{\frac{1}{2}}^{1}\|\nabla^i b(\cdot,1-s)\|_{L^1(\mathbb{R}^n)}\,ds\right)\\
&\leq C\|f\|_{Y_1^1}\int_0^{\frac{1}{2}} s^{-\frac{i}{4}}\,ds\\
&\leq C\|f\|_{Y_1^1}.
\end{aligned}
$$

$$
\begin{aligned}
|I_2| &\leq (\sup_{0\leq s\leq\frac{1}{2}}\|\nabla^i b(\cdot,1-s)\|_{L^\infty(\mathbb{R}^n)})(\int_{B_2\times[0,\frac{1}{2}]}|f(y,s)|\,dyds)\\
&\leq C\int_{B_2\times[0,\frac{1}{2}]}|f(y,s)|\,dyds\leq C\|f\|_{Y_1^1},
\end{aligned}
$$

and

$$
\begin{aligned}
|I_3| &\leq \int_0^{\frac{1}{2}}\int_{\mathbb{R}^n\setminus B_2}|\nabla^i b(y,1-s)||f(y,s)|\,dyds\\
&\leq C\int_0^{\frac{1}{2}}\int_{\mathbb{R}^n\setminus B_2}e^{-c_1|y|}|f(y,s)|\,dyds\\
&\leq C\left(\sum_{k=2}^{\infty}k^{n-1}e^{-c_1 k}\right)\cdot\left(\sup_{y\in\mathbb{R}^n}\int_{P_1(y,1)}|f(y,s)|\,dyds\right)\\
&\leq C\|f\|_{Y_1^1}.
\end{aligned}
$$

Now we want to show (4.5) by the energy method. Denote $w=\mathbb{S}f$. Then w solves

$$
(\partial_t+\Delta^2)w=f\quad\text{in }\mathbb{R}^n\times(0,+\infty);\;\; w|_{t=0}=0. \tag{4.6}
$$

Let $\eta\in C_0^\infty(B_2)$ be a cut-off function of B_1. Multiplying (4.6) by $\eta^4 w$ and integrating over $\mathbb{R}^n\times[0,1]$, we obtain

$$
\int_{\mathbb{R}^n\times\{1\}}|w|^2\eta^4+2\int_{\mathbb{R}^n\times[0,1]}\Delta w\cdot\Delta(w\eta^4)=\int_{\mathbb{R}^n\times[0,1]}f\cdot w\eta^4.
$$

This easily implies

$$
\begin{aligned}
& \int_{P_1(0,1)} |\nabla^2 w|^2 \\
\leq\ & \int_{\mathbb{R}^n\times[0,1]} |\nabla^2(\eta^2 w)|^2 \\
\leq\ & C\int_{\mathbb{R}^n\times[0,1]} \left[|\nabla\eta|^2|\nabla w|^2 + (|\Delta\eta| + |\nabla\eta|^2)|w|^2\right] + C\int_{\mathbb{R}^n\times[0,1]} |f||w|\eta^2 \\
\leq\ & C[\int_{(B_2\setminus B_1)\times[0,1]} |\nabla w|^2 + |w|^2 + \|f\|_{L^1(B_2\times[0,1])}\|w\|_{L^\infty(B_2\times[0,1])}] \\
\leq\ & C\left[(\int_0^1 t^{\frac{1}{2}}\,dt)\cdot(\sup_{0<t\leq 1} t^{\frac{1}{2}}\|\nabla w(t)\|^2_{L^\infty(\mathbb{R}^n)}) + \|w\|^2_{L^\infty(B_2\times[0,1])} + \|f\|^2_{L^1(B_2\times[0,1])}\right] \\
\leq\ & C\left[\sup_{0<t\leq 1}(\|w(t)\|^2_{L^\infty(\mathbb{R}^n)} + t^{\frac{1}{2}}\|\nabla w(t)\|^2_{L^\infty(\mathbb{R}^n)}) + \|f\|^2_{Y^1_1}\right] \\
\leq\ & C\|f\|^2_{Y^1_1},
\end{aligned}
\tag{4.7}
$$

where we have used (4.4) in the last step.

For the L^4 norm of ∇w on $P_1(0,1)$, recall the Nirenberg inequality implies

$$\|\nabla(\eta^2 w(t))\|^4_{L^4(\mathbb{R}^n)} \leq C\|\eta^2 w(t)\|^2_{L^\infty(\mathbb{R}^n)}\|\nabla^2(\eta^2 w(t))\|^2_{L^2(\mathbb{R}^n)}.$$

Integrating with respect to $t\in[0,1]$ clearly implies

$$\left(\int_{P_1(0,1)} |\nabla w|^4\right)^{\frac{1}{4}} \leq C\sup_{0\leq t\leq 1}\|w(t)\|^{\frac{1}{2}}_{L^\infty(\mathbb{R}^n)}\|\nabla^2(\eta^2 w)\|^{\frac{1}{2}}_{L^2(\mathbb{R}^n\times[0,1])} \leq C\|f\|_{Y^1_1},$$

where we have used both (4.4) and (4.7) in the last step. This completes the proof. □

To handle the nonlinearities of the heat flow of biharmonic maps (1.3), we also need

Lemma 4.2 *For $0 < T \leq +\infty$, if $f\in Y^2_T$, then for any $1\leq\alpha\leq n$, $\mathbb{S}(\frac{\partial f}{\partial x_\alpha})\in X_T$ and*

$$\left\|\mathbb{S}(\frac{\partial f}{\partial x_\alpha})\right\|_{X_T} \leq C\,\|f\|_{Y^2_T} \tag{4.8}$$

for some $C = C(n) > 0$.

Proof. The proof of (4.8) is similar to that of Lemma 4.1. We will prove that for any $x\in\mathbb{R}^n$ and $0 < R\leq T^{\frac{1}{4}}$, both the pointwise estimate:

$$\sum_{i=0}^{2} R^i\left|\nabla^i(\mathbb{S}(\frac{\partial f}{\partial x_\alpha}))\right|(x,R^4) \leq C\|f\|_{Y^2_T}, \tag{4.9}$$

and the integral estimate:

$$R^{-\frac{n}{4}}\left\|\nabla(\mathbb{S}(\frac{\partial f}{\partial x_\alpha}))\right\|_{L^4(P_R(x,R^4))}+R^{-\frac{n}{2}}\left\|\nabla^2(\mathbb{S}(\frac{\partial f}{\partial x_\alpha}))\right\|_{L^2(P_R(x,R^4))}\le C\|f\|_{Y^2_T}. \quad (4.10)$$

By suitable scalings, we assume $T\ge 1$. Since both estimates are translation and scale invariant, it suffices to show that both (4.9) and (4.10) hold for $x=0$ and $R=1$. For $1\le\alpha\le n$, write $W_\alpha=\mathbb{S}(\frac{\partial f}{\partial x_\alpha})$. For $i=0,1,2$, we have

$$\begin{aligned}\nabla^i W_\alpha(0,1) &= \int_{\mathbb{R}^n\times[0,1]}\nabla^i b(-y,1-s)\frac{\partial f}{\partial y_\alpha}(y,s)\,dyds\\ &= \int_{\mathbb{R}^n\times[0,1]}(\nabla^i\frac{\partial}{\partial y_\alpha}b)(-y,1-s)f(y,s)\,dyds,\end{aligned}$$

which implies

$$\begin{aligned}\left|\nabla^i W_\alpha(0,1)\right| &\le \int_0^1\int_{\mathbb{R}^n}|\nabla^{i+1}b(y,1-s)||f(y,s)|\,dyds\\ &= \left\{\int_{\frac12}^1\int_{\mathbb{R}^n}+\int_0^{\frac12}\int_{B_2}+\int_0^{\frac12}\int_{\mathbb{R}^n\setminus B_2}\right\}|\nabla^{i+1}b(y,1-s)||f(y,s)|\,dyds\\ &= I_4+I_5+I_6.\end{aligned}$$

Applying Lemma 2.1, we can estimate I_4,I_5,I_6 as follows.

$$\begin{aligned}|I_4| &\le \left(\sup_{\frac12\le s\le 1}\|f(s)\|_{L^\infty(\mathbb{R}^n)}\right)\left(\int_{\frac12}^1\|\nabla^{i+1}b(\cdot,1-s)\|_{L^1(\mathbb{R}^n)}\,ds\right)\\ &\le C\|f\|_{Y^2_1}\int_0^{\frac12}s^{-\frac{i+1}{4}}\,ds\\ &\le C\|f\|_{Y^2_1},\end{aligned}$$

where we have used the fact $\int_0^{\frac12}s^{-\frac{i+1}{4}}\,ds<+\infty$ for $i\le 2$.

$$\begin{aligned}|I_5| &\le (\sup_{0\le s\le\frac12}\|\nabla^{i+1}b(\cdot,1-s)\|_{L^\infty(\mathbb{R}^n)})(\int_{B_2\times[0,\frac12]}|f(y,s)|\,dyds)\\ &\le C\int_{B_2\times[0,\frac12]}|f(y,s)|\,dyds\le C\|f\|_{Y^2_1},\end{aligned}$$

and since $i+1\le 3$, we have

$$\begin{aligned}|I_6| &\le \int_0^{\frac12}\int_{\mathbb{R}^n\setminus B_2}|\nabla^{i+1}b(y,1-s)||f(y,s)|\,dyds\\ &\le C\int_0^{\frac12}\int_{\mathbb{R}^n\setminus B_2}e^{-c_1|y|}|f(y,s)|\,dyds\\ &\le C\left(\sum_{k=2}^\infty k^{n-1}e^{-c_1k}\right)\cdot\left(\sup_{y\in\mathbb{R}^n}\int_{P_1(y,1)}|f(y,s)|\,dyds\right)\\ &\le C\|f\|_{Y^2_1}.\end{aligned}$$

Putting together these estimates, we prove (4.9). (4.10) can be done by the energy method as well. In fact, W_α solves

$$(\partial_t + \Delta^2)W_\alpha = \frac{\partial f}{\partial x_\alpha} \ \text{ in } \mathbb{R}^n \times (0, +\infty);\ W_\alpha|_{t=0} = 0. \tag{4.11}$$

Let $\eta \in C_0^\infty(B_2)$ be a cut-off function of B_1. Multiplying (4.11) by $\eta^4 W_\alpha$ and integrating over $\mathbb{R}^n \times [0,1]$, we obtain

$$\int_{\mathbb{R}^n\times\{1\}} |W_\alpha|^2\eta^4 + 2\int_{\mathbb{R}^n\times[0,1]} \Delta W_\alpha \cdot \Delta(W_\alpha\eta^4) = -\int_{\mathbb{R}^n\times[0,1]} f \cdot \frac{\partial}{\partial x_\alpha}(W_\alpha\eta^4).$$

This implies

$$\begin{aligned}
& \int_{P_1(0,1)} |\nabla^2 W_\alpha|^2 \\
\leq\ & \int_{\mathbb{R}^n\times[0,1]} |\nabla^2(\eta^2 W_\alpha)|^2 \\
\leq\ & C\int_{\mathbb{R}^n\times[0,1]} \left[|\nabla\eta|^2|\nabla W_\alpha|^2 + (|\Delta\eta| + |\nabla\eta|^2)|W_\alpha|^2\right] \\
+\ & C\int_{\mathbb{R}^n\times[0,1]} |f|(|\nabla(\eta^2 W_\alpha)| + |W_\alpha||\nabla\eta|) \\
\leq\ & C[\int_{(B_2\setminus B_1)\times[0,1]} (|\nabla W_\alpha|^2 + |W_\alpha|^2) + \|f\|_{L^1(B_2\times[0,1])}\|W_\alpha\|_{L^\infty(\mathbb{R}^n)}] \\
+\ & C\|f\|_{L^{\frac{4}{3}}(B_2\times[0,1])}\|\nabla(\eta^2 W_\alpha)\|_{L^4(\mathbb{R}^n\times[0,1])} \\
=\ & I_7 + I_8.
\end{aligned} \tag{4.12}$$

It is easy to see that

$$\begin{aligned}
& |I_7| \\
\leq\ & C\left[(\int_0^1 t^{\frac{1}{2}}\,dt)\cdot(\sup_{0<t\leq 1} t^{\frac{1}{2}}\|\nabla W_\alpha(t)\|^2_{L^\infty(\mathbb{R}^n)}) + \|W_\alpha\|^2_{L^\infty(B_2\times[0,1])} + \|f\|^2_{L^1(B_2\times[0,1])}\right] \\
\leq\ & C\|f\|^2_{Y_1^2}
\end{aligned}$$

where we have used the point wise estimate (4.9) in the last step. In order to estimate I_8, we first need to employ the Nirenberg inequality:

$$\|\nabla(\eta^2 W_\alpha(t))\|^4_{L^4(\mathbb{R}^n)} \leq C\|\eta^2 W_\alpha(t)\|^2_{L^\infty(\mathbb{R}^n)}\|\nabla^2(\eta^2 W_\alpha(t))\|^2_{L^2(\mathbb{R}^n)},$$

which, after integrating with respect to $t \in [0,1]$, implies

$$\|\nabla(\eta^2 W_\alpha)\|_{L^4(\mathbb{R}^n\times[0,1])} \leq C\sup_{0\leq t\leq 1}\|W_\alpha(t)\|^{\frac{1}{2}}_{L^\infty(\mathbb{R}^n)}\|\nabla^2(\eta^2 W_\alpha)\|^{\frac{1}{2}}_{L^2(\mathbb{R}^n\times[0,1])}. \tag{4.13}$$

Therefore, I_8 can be estimated by

$$\begin{aligned}
|I_8| &\le C\|f\|_{L^{\frac{4}{3}}(B_2\times[0,1])} \sup_{0\le t\le 1}\|W_\alpha(t)\|_{L^\infty(\mathbb{R}^n)}^{\frac{1}{2}}\|\nabla^2(\eta^2 W_\alpha)\|_{L^2(\mathbb{R}^n\times[0,1])}^{\frac{1}{2}}\\
&\le \frac{1}{2}\int_{\mathbb{R}^n\times[0,1]}|\nabla^2(\eta^2W_\alpha)|^2 + C\|f\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(B_2\times[0,1])}\sup_{0\le t\le 1}\|W_\alpha(t)\|^{\frac{2}{3}}_{L^\infty(\mathbb{R}^n)}\\
&\le \frac{1}{2}\int_{\mathbb{R}^n\times[0,1]}|\nabla^2(\eta^2W_\alpha)|^2 + C\|f\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(B_2\times[0,1])}\sup_{0\le t\le 1}\|W_\alpha(t)\|^{\frac{2}{3}}_{L^\infty(\mathbb{R}^n)}\\
&\le \frac{1}{2}\int_{\mathbb{R}^n\times[0,1]}|\nabla^2(\eta^2W_\alpha)|^2 + C\|f\|^2_{Y_1^2},
\end{aligned}$$

where we have used (4.9) in the last step. Now we substitute the estimates of I_7 and I_8 into (4.12) and obtain

$$\int_{P_1(0,1)}|\nabla^2 W_\alpha|^2 \le C\int_{\mathbb{R}^n\times[0,1]}|\nabla^2(\eta^2W_\alpha)|^2 \le C\|f\|^2_{Y_1^2}.$$

This, combined with (4.13), also implies

$$\int_{P_1(0,1)}|\nabla W_\alpha|^4 \le C\|f\|^4_{Y_1^2}.$$

The proof of (4.10) is now completed. □

5 Proof of Theorem 1.2 and 1.3

In this section, we will prove both Theorem 1.2 and 1.3. The idea is based on the fixed point theorem in a small ball inside X_T for the mapping operator determined by the Duhamel formula associate with (1.3)-(1.4).

First we need to extend Π to $\mathbb{R}^l$. Let $\widetilde{\Pi}\in C^\infty(\mathbb{R}^l,\mathbb{R}^l)$ be any smooth extension of Π such that $\widetilde{\Pi}\equiv\Pi$ on N_{δ_N}.

Let

$$\mathcal{F}[u]=\Delta(D^2\widetilde{\Pi}(u)(\nabla u,\nabla u))-\langle\Delta u,\Delta(D\widetilde{\Pi}(u))\rangle+2\nabla\cdot\langle\Delta u,\nabla(D\widetilde{\Pi}(u)))\rangle$$

be the right hand side nonlinearity of (1.3). Then it is easy to see that

$$\begin{aligned}
\mathcal{F}[u] &= -\langle\Delta u,\Delta(D\widetilde{\Pi}(u))\rangle+\nabla\cdot\Big(2\langle\Delta u,\nabla(D\widetilde{\Pi}(u))\rangle+\nabla(D^2\widetilde{\Pi}(u)(\nabla u,\nabla u))\Big)\\
&= \mathcal{F}_1[u]+\nabla\cdot(\mathcal{F}_2[u]), \qquad (5.1)
\end{aligned}$$

where

$$\mathcal{F}_1[u] = -\langle \Delta u, \Delta(D\widetilde{\Pi}(u))\rangle,\ \mathcal{F}_2[u] = 2\langle \Delta u, \nabla(D\widetilde{\Pi}(u))\rangle + \nabla(D^2\widetilde{\Pi}(u)(\nabla u, \nabla u)). \tag{5.2}$$

It is easy to see

$$|\mathcal{F}_1[u]| \le C(|\nabla^2 u|^2 + |\nabla u|^4),\ |\mathcal{F}_2[u]| \le C(|\nabla^2 u||\nabla u| + |\nabla u|^3), \tag{5.3}$$

where $C > 0$ is a constant depending on $\|u\|_{L^\infty(\mathbb{R}^n)}$. With the notations as above, (1.3)-(1.4) can be written as

$$(\partial_t + \Delta^2)u = \mathcal{F}_1[u] + \nabla\cdot(\mathcal{F}_2[u]) \ \text{ in } \mathbb{R}^n\times(0,+\infty);\ u|_{t=0} = u_0. \tag{5.4}$$

The first observation is

Lemma 5.1 *For $0 < T \le +\infty$, if $u \in X_T$, then $\mathcal{F}_1[u] \in Y_T^1, \mathcal{F}_2[u] \in Y_T^2$. Moreover,*

$$\|\mathcal{F}_1[u]\|_{Y_T^1} \le C\,[u]_{X_T}^2\,, \tag{5.5}$$

and

$$\|\mathcal{F}_2[u]\|_{Y_T^2} \le C\,[u]_{X_T}^2\,, \tag{5.6}$$

Proof. It follows directly from the Hölder inequality. □

By the Duhamel formula (2.10), the solution u to (1.3)-(1.4) is given by

$$u = \mathbb{G}u_0 + \mathbb{S}(\mathcal{F}_1[u]) + \mathbb{S}(\nabla\cdot(\mathcal{F}_2[u])). \tag{5.7}$$

Throughout this section, we denote

$$\hat{u}_0 = \mathbb{G}u_0.$$

Now we define the mapping operator $\mathbb{T}$ on X_{R^4} by letting

$$\mathbb{T}u(x,t) = \hat{u}_0(x,t) + \mathbb{S}(\mathcal{F}_1[u])(x,t) + \mathbb{S}(\nabla\cdot(\mathcal{F}_2[u]))(x,t), \quad u \in X_{R^4}. \tag{5.8}$$

The following property follows directly from Lemma 3.1.

Lemma 5.2 *For any $R > 0$ and any initial map $u_0 : \mathbb{R}^n \to N$, $\hat{u}_0 \in X_{R^4}$ and*

$$\|\hat{u}_0\|_{L^\infty(\mathbb{R}^{n+1}_+)} \le C\|u_0\|_{L^\infty(\mathbb{R}^n)}, \quad [\hat{u}_0]_{X_{R^4}} \le C\,[u_0]_{\mathrm{BMO_R}(\mathbb{R}^n)}\,. \tag{5.9}$$

For $\epsilon > 0$, we define

$$\mathbb{B}_\epsilon(\hat{u}_0) := \left\{u \in X_{R^4} : \ \|u - \hat{u}_0\|_{X_{R^4}} \leq \epsilon\right\}$$

to be the ball in X_{R^4} with center $\hat{u}_0$ and radius ϵ. By the triangle inequality, we have

$$\|u\|_{L^\infty(\mathbb{R}^{n+1}_+)} \leq \epsilon + C\|u_0\|_{L^\infty(\mathbb{R}^n)}, \quad [u]_{X_{R^4}} \leq \epsilon + [u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)}, \ \forall u \in \mathbb{B}_\epsilon(\hat{u}_0). \tag{5.10}$$

In particular, we have

Lemma 5.3 *For $0 < R \leq +\infty$, if $u_0 : \mathbb{R}^n \to N$ has $[u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)} \leq \epsilon$, then*

$$\|u\|_{L^\infty(\mathbb{R}^{n+1}_+)} \leq C + \epsilon, \quad [u]_{X_{R^4}} \leq C\epsilon, \ \forall u \in \mathbb{B}_\epsilon(\hat{u}_0) \tag{5.11}$$

for some $C = C(n, N) > 0$.

The proof of Theorem 1.2 is based on the following two lemmas.

Lemma 5.4 *There exists $\epsilon_1 > 0$ such that for any $0 < R \leq +\infty$ if $u_0 : \mathbb{R}^n \to N$ has*

$$[u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)} \leq \epsilon_1,$$

then $\mathbb{T}$ maps $\mathbb{B}_{\epsilon_1}(\hat{u}_0)$ to $\mathbb{B}_{\epsilon_1}(\hat{u}_0)$.

Proof. By (5.8), we have

$$\mathbb{T}(u) - \hat{u}_0 = \mathbb{S}(\mathcal{F}_1[u]) + \mathbb{S}(\nabla \cdot (\mathcal{F}_2[u])), \quad u \in \mathbb{B}_{\epsilon_1}(\hat{u}_0).$$

Hence Lemma 4.1, Lemma 4.2, Lemma 5.1, and Lemma 5.2 imply that for any $u \in \mathbb{B}_{\epsilon_1}(\hat{u}_0)$,

$$\begin{aligned}
&\|\mathbb{T}(u) - \hat{u}_0\|_{X_{R^4}} \\
\lesssim\ & \|\mathbb{S}(\mathcal{F}_1[u])\|_{X_{R^4}} + \|\mathbb{S}(\nabla \cdot (\mathcal{F}_2[u]))\|_{X_{R^4}} \\
\lesssim\ & \|\mathcal{F}_1[u]\|_{Y^1_{R^4}} + \|\mathcal{F}_2[u]\|_{Y^2_{R^4}} \\
\lesssim\ & [u]^2_{X_{R^4}} \leq C\epsilon_1^2 \leq \epsilon_1,
\end{aligned}$$

provided $\epsilon_1 > 0$ is chosen to be sufficiently small. Hence $\mathbb{T}u \in \mathbb{B}_{\epsilon_1}(\hat{u}_0)$. This completes the proof. □

Lemma 5.5 *There exist $0 < \epsilon_2 \le \epsilon_1$ and $\theta_0 \in (0,1)$ such that for $0 < R \le +\infty$ if $u_0 : \mathbb{R}^n \to N$ satisfies*

$$[u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)} \le \epsilon_2$$

then $\mathbb{T} : \mathbb{B}_{\epsilon_2}(\hat{u}_0) \to \mathbb{B}_{\epsilon_2}(\hat{u}_0)$ is a θ_0-contraction map, i.e.

$$\|\mathbb{T}(u) - \mathbb{T}(v)\|_{X_{R^4}} \le \theta_0 \|u - v\|_{X_{R^4}}, \ \forall u, v \in \mathbb{B}_{\epsilon_2}(\hat{u}_0).$$

Proof. For $u, v \in \mathbb{B}_{\epsilon_2}(\hat{u}_0)$, we have

$$\begin{aligned}
\|\mathbb{T}u - \mathbb{T}v\|_{X_{R^4}} &\le \|\mathbb{S}(\mathcal{F}_1[u] - \mathcal{F}_1[v])\|_{X_{R^4}} + \|\mathbb{S}(\nabla \cdot (\mathcal{F}_2[u] - \mathcal{F}_2[v]))\|_{X_{R^4}} \\
&\lesssim \|\mathcal{F}_1[u] - \mathcal{F}_1[v]\|_{Y^1_{R^4}} + \|\mathcal{F}_2[u] - \mathcal{F}_2[v]\|_{Y^2_{R^4}}. \qquad (5.12)
\end{aligned}$$

Since

$$\begin{aligned}
\mathcal{F}_1[u] - \mathcal{F}_1[v] &= -\langle \Delta u, \Delta(D\widetilde{\Pi}(u))\rangle + \langle \Delta v, \Delta(D\widetilde{\Pi}(v))\rangle \\
&= \langle \Delta(u - v), \Delta(D\widetilde{\Pi}(u))\rangle + \langle \Delta v, \Delta(D\widetilde{\Pi}(u) - D\widetilde{\Pi}(v))\rangle,
\end{aligned}$$

we have

$$\begin{aligned}
|\mathcal{F}_1[u] - \mathcal{F}_1[v]| &\le C[|\Delta(u - v)|(|\Delta u| + |\nabla u|^2 + |\Delta v|) \\
&+ |\Delta v|(|\nabla u| + |\nabla v|)|\nabla(u - v)|)] + C|\Delta v|(|\nabla^2 u| + |\nabla^2 v|)|u - v|.
\end{aligned}$$

Hence

$$\begin{aligned}
\|\mathcal{F}_1[u] - \mathcal{F}_1[v]\|_{Y^1_{R^4}} &\le C[([u]_{X_{R^4}} + [v]_{X_{R^4}} + [u]^2_{X_{R^4}})\|u - v\|_{X_{R^4}} \\
&\quad + [v]_{X_{R^4}}([u]_{X_{R^4}} + [v]_{X_{R^4}})\|u - v\|_{X_{R^4}}] \\
&\le C\epsilon_2 \|u - v\|_{X_{R^4}}, \qquad (5.13)
\end{aligned}$$

where we have used Lemma 5.3 in the last step.

Since

$$\begin{aligned}
|\mathcal{F}_2[u] - \mathcal{F}_2[v]| &\le |2(\langle \Delta u, \nabla(D\widetilde{\Pi}(u))\rangle - \langle \Delta v, \nabla(D\widetilde{\Pi}(v))\rangle)| \\
&+ |\nabla(D^2\widetilde{\Pi}(u)(\nabla u, \nabla u) - D^2\widetilde{\Pi}(v)(\nabla v, \nabla v))| \\
&\le C[|\nabla u||\Delta(u - v)| + |\Delta v|(|u - v| + |\nabla(u - v))|] \\
&+ C[|\nabla u|(|\nabla u| + |\nabla v|)|\nabla(u - v)| + (|\nabla^2 u| + |\nabla^2 v|)|\nabla(u - v)|] \\
&+ C[(|\nabla u| + |\nabla v|)|\nabla^2(u - v)| + |\nabla v|^2|\nabla(u - v)| + |\nabla v||\nabla^2 v||u - v|],
\end{aligned}$$

we have

$$\begin{aligned}\|\mathcal{F}_2[u]-\mathcal{F}_2[v]\|_{Y^2_{R^4}} &\le C([u]_{X_{R^4}}+[v]_{X_{R^4}}+[u]^2_{X_{R^4}}+[v]^2_{X_{R^4}})\|u-v\|_{X_{R^4}}\\ &\le C\epsilon_2\|u-v\|_{X_{R^4}}. \qquad (5.14)\end{aligned}$$

Putting (5.13) and (5.14) into (5.12), we obtain

$$\|\mathbb{T}u-\mathbb{T}v\|_{X_{R^4}} \le C\epsilon_2\|u-v\|_{X_{R^4}} \le \theta_0\|u-v\|_{X_{R^4}}$$

for some $\theta_0=\theta_0(\epsilon_2)\in(0,1)$, provided $\epsilon_2>0$ is chosen to be sufficiently small. This completes the proof. $\square$

Proof of Theorem 1.2. It follows from Lemma 5.4 and Lemma 5.5, and the fixed point theorem that there exists $\epsilon_0>0$ such that for $0<R\le+\infty$ if $[u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)}\le\epsilon_0$, then there exists a unique $u\in X_{R^4}$ such that

$$u=\hat{u}_0+\mathbb{S}(\mathcal{F}[u]) \ \text{ on } \mathbb{R}^n\times[0,R^4),$$

or equivalently

$$u_t+\Delta^2u=\mathcal{F}[u] \text{ on } \mathbb{R}^n\times(0,R^4);\ u\big|_{t=0}=u_0.$$

Now we need to show $u(\mathbb{R}^n\times[0,R^4])\subset N$. First, observe that Lemma 2.1 implies that for any $x\in\mathbb{R}^n$ and $t\le\frac{R^4}{K^4}$,

$$\begin{aligned}\mathrm{dist}(u(x,t),N) &\le \mathrm{dist}(\hat{u}_0(x,t),N)+\|u-\hat{u}_0\|_{L^\infty(\mathbb{R}^n\times[0,R^4])}\\ &\le \delta+K^n\,[u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)}+\epsilon_0\\ &\le \delta+(1+K^n)\epsilon_0\le\delta_N,\end{aligned}$$

provide $\delta\le\frac{\delta_N}{2}$ and $\epsilon_0\le\frac{\delta_N}{2(1+K^n)}$. This yields $u(\mathbb{R}^n\times[0,\frac{R^4}{K^4}])\subset N_{\delta_N}$, the δ_N-neighborhood of N. This and the definition of $\widetilde{\Pi}(\cdot)$ imply that $\widetilde{\Pi}(u)\equiv\Pi(u)$.

Set $Q(y)=y-\Pi(y)$ for $y\in N_{\delta_N}$, and $\rho(u)=\frac{1}{2}|Q(u)|^2$. Then direct calculations imply that for any $y\in N_{\delta_N}$,

$$DQ(y)(v)=(\mathrm{Id}-D\Pi(y))(v),\ \forall v\in\mathbb{R}^l,$$

and

$$D^2Q(y)(v,w) = -D^2\Pi(y)(v,w),\ \forall v,w \in \mathbb{R}^l.$$

Observe that $\mathcal{F}[u]$ can be rewritten as

$$\begin{aligned}
&\mathcal{F}[u] \\
= \quad &\Delta(D^2\Pi(u)(\nabla u, \nabla u)) + \nabla\cdot(D^2\Pi(u)(\Delta u, \nabla u)) + D^2\Pi(u)(\nabla\Delta u, \nabla u).
\end{aligned}$$

Direct calculations imply

$$\begin{aligned}
&(\partial_t + \Delta^2)Q(u) \\
= \quad & DQ(u)(\partial_t u + \Delta^2 u) \\
- \quad & \left[D^2\Pi(u)(\nabla\Delta u, \nabla u) + \nabla\cdot(D^2\Pi(u)(\Delta u, \nabla u)) + \Delta(D^2\Pi(u)(\nabla u, \nabla u))\right] \\
= \quad & DQ(u)(\mathcal{F}[u]) - \mathcal{F}[u] \\
= \quad & -D\Pi(u)(\mathcal{F}[u]). \qquad (5.15)
\end{aligned}$$

Multiplying both sides of (5.15) by $Q(u)$ and integrating over $\mathbb{R}^n$, we obtain

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^n}\rho(u) + \frac{1}{2}\int_{\mathbb{R}^n}|\Delta(Q(u))|^2 \quad &= \quad -\frac{1}{2}\int_{\mathbb{R}^n}\langle D\Pi(u)(\mathcal{F}[u]), Q(u)\rangle \\
&= \quad 0, \qquad (5.16)
\end{aligned}$$

where we have used the fact that $Q(u) \perp T_{\Pi(u)}N$ and $D\Pi(u)(\mathcal{F}[u]) \in T_{\Pi(u)}N$ in the last step.

Since $\rho(u)|_{t=0} = 0$, integrating (5.16) from 0 to $\frac{R^4}{K^4}$ implies $\rho(u) \equiv 0$ on $\mathbb{R}^n \times [0, \frac{R^4}{K^4}]$. Thus $u(\mathbb{R}^n \times [0, \frac{R^4}{K^4}]) \subset N$. Repeating the same argument for $t \in [\frac{R^4}{K^4}, R^4]$ yields $u(\mathbb{R}^n \times [\frac{R^4}{K^4}, R^4]) \subset N$. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. It follows directly from Theorem 1.2 with $R = +\infty$. □

6 Proof of Theorem 1.4 and 1.5

This section is devoted to the proof of both Theorem 1.4 and 1.5. Since the argument is similar to that of Theorem 1.2, we will only sketch it here.

Let $\mathcal{H}[u]$ denote the right hand side of (1.7). Then we have

$$\mathcal{H}[u] = \mathcal{F}_1[u] + \nabla \cdot \mathcal{F}_2[u] + \mathcal{F}_3[u],$$

where $\mathcal{F}_1[u]$ and $\mathcal{F}_2[u]$ are given by (5.2), while

$$\begin{aligned} \mathcal{F}_3[u] &= D\widetilde{\Pi}(u)[D^2\widetilde{\Pi}(u)(\nabla u, \nabla u) \cdot D^3\widetilde{\Pi}(u)(\nabla u, \nabla u)] \\ &\quad +2D^2\widetilde{\Pi}(u)(\nabla u, \nabla u) \cdot D^2\widetilde{\Pi}(u)(\nabla u, \nabla(D\widetilde{\Pi}(u))). \end{aligned} \tag{6.1}$$

It is clear that $u \in X_{R^4}$ solves (1.7)-(1.8) iff

$$u = \mathbb{G}u_0 + \mathbb{S}(\mathcal{F}_1[u]) + \mathbb{S}(\nabla \cdot \mathcal{F}_2[u]) + \mathbb{S}(\mathcal{F}_3[u]). \tag{6.2}$$

Since $\mathcal{F}_3[u]$ satisfies

$$|\mathcal{F}_3[u]| \le C|\nabla u|^4, \tag{6.3}$$

for some $C > 0$ depending on $\|u\|_{L^\infty(\mathbb{R}^n)}$, it is easy to check

Claim 1. *For $0 < R \le +\infty$, if $u \in X_{R^4}$, then $\mathcal{F}_3[u] \in Y^1_{R^4}$ and*

$$\|\mathcal{F}_3[u]\|_{Y^1_{R^4}} \le C\,[u]^4_{X_{R^4}}\,. \tag{6.4}$$

This claim and Lemma Lemma 4.1 then imply

Claim 2. *For $0 < R \le +\infty$, if $u \in X_{R^4}$, then $\mathbb{S}(\mathcal{F}_3[u]) \in X_{R^4}$ and*

$$\|\mathbb{S}(\mathcal{F}_3[u])\|_{X_{R^4}} \le C\,[u]^4_{X_{R^4}}\,. \tag{6.5}$$

Now if define the mapping operator $\widetilde{\mathbb{T}}$ on X_{R^4} by

$$\widetilde{\mathbb{T}}[u] := \mathbb{G}u_0 + \mathbb{S}(\mathcal{F}_1[u]) + \mathbb{S}(\nabla \cdot \mathcal{F}_2[u]) + \mathbb{S}(\mathcal{F}_3[u]), \tag{6.6}$$

then Claim 1, Claim 2, and Lemma 5.4 imply

Claim 3. *There exists $\epsilon_3 > 0$ such that for $0 < R \le +\infty$, if $u_0 : \mathbb{R}^n \to N$ has $[u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)} \le \epsilon_3$, then $\widetilde{\mathbb{T}}$ maps $\mathbb{B}_{\epsilon_3}(\hat{u}_0)$ to $\mathbb{B}_{\epsilon_3}(\hat{u}_0)$.*

We need to show $\widetilde{\mathbb{T}} : \mathbb{B}_{\epsilon_3}(\hat{u}_0) \to \mathbb{B}_{\epsilon_3}(\hat{u}_0)$ is a contraction map. To see this, observe that direct calculations imply that for any $u, v \in \mathbb{B}_{\epsilon_3}(\hat{u}_0)$,

$$\begin{aligned} &|\mathcal{F}_3[u] - \mathcal{F}_3[v]| \qquad (6.7) \\ \le\ & C[|u - v||\nabla u|^4 + |\nabla(u - v)|(|\nabla v|^3 + |\nabla v||\nabla u|^2 + |\nabla v|^2|\nabla u| + |\nabla u|^3)] \end{aligned}$$

for some $C > 0$ depending only $\max\{\|u\|_{L^\infty(\mathbb{R}^n)}, \|v\|_{L^\infty(\mathbb{R}^n)}\}$. Hence, combined with the proof of Lemma 5.5, we obtain

Claim 4. *There exists $\epsilon_3 > 0$ such that for $0 < R \leq +\infty$, if $u_0 : \mathbb{R}^n \to N$ has $[u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)} \leq \epsilon_3$, then*

$$\|\widetilde{\mathbb{T}}[u] - \widetilde{\mathbb{T}}[v]\|_{X_{R^4}} \leq C\epsilon_3 \|u - v\|_{X_{R^4}}, \ \forall u, v \in \mathbb{B}_{\epsilon_3}(\hat{u}_0). \tag{6.8}$$

Now we can complete the proof of Theorem 1.4 as follows.

Completion of proof of Theorem 1.4: Similar to Theorem 1.2, it follows from Claim 3 and Claim 4 and the fixed point theorem that there exists $\epsilon_0 > 0$ such that for $0 < R \leq +\infty$ if $[u_0]_{\mathrm{BMO}_R(\mathbb{R}^n)} \leq \epsilon_0$, then there exists a unique $u \in X_{R^4}$ that solves (1.7)-(1.8):

$$u_t + \Delta^2 u = \mathcal{H}[u] \text{ on } \mathbb{R}^n \times (0, R^4); \ u\big|_{t=0} = u_0.$$

The same argument as in Theorem 1.2 implies $u(\mathbb{R}^n \times [0, \frac{R^4}{K^4}]) \subset N_{\delta_N}$. Hence $\widetilde{\Pi}(u) \equiv \Pi(u)$ on $\mathbb{R}^n \times [0, \frac{R^4}{K^4}]$. Moreover, the same calculation as in (5.15) implies

$$(\partial_t + \Delta^2)(u - D\Pi(u)) = -D\Pi(u)(\mathcal{H}[u]), \tag{6.9}$$

and it follows that for $0 \leq t \leq \frac{R^4}{K^4}$,

$$\frac{d}{dt}\int_{\mathbb{R}^n} |u - D\Pi(u)|^2 + \int_{\mathbb{R}^n} |\Delta(u - D\Pi(u))|^2 = 0. \tag{6.10}$$

This, combined with $|u - D\Pi(u)|^2\big|_{t=0} = 0$, implies that $u(\mathbb{R}^n \times [0, \frac{R^4}{K^4}]) \subset N$. Repeating the same argument then implies $u(\mathbb{R}^n \times [0, R^4]) \subset N$. The proof is complete.
□

Proof of Theorem 1.5. It follows directly from Theorem 1.4 with $R = +\infty$. □

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